ON STRONG AVERAGES OF SPHERICAL FOURIER SUMS

O.I.KUZNETSOVA, A.N.PODKORYTOV

We study two similar summation methods for multiple Fourier series which are analogs of the classical strong summation method introduced by Hardy and Littlewood [1] in the one-dimensional case.

Let

$$S_R(f,x)=\sum_{\|k\|\leqslant R}\widehat{f}(k)\,e^{ik\cdot x},\quad \text{where}\ \widehat{f}(k)=\frac{1}{(2\pi)^m}\int_{\mathbb{T}^m}f(u)\,e^{-ik\cdot u}du\quad \text{for}\ \ k\in\mathbb{Z}^m,$$

be a spherical partial sum for the Fourier series of a function f which is integrable on the cube $\mathbb{T}^m=[-\pi,\pi]^m$. We say that the sequence $\{S_n(f,x)\}$ is H_p-summable (in what follows, $p\geqslant 1$ is a fixed parameter) to the sum S if

$$\frac{1}{n}\sum_{j=0}^{n-1}|S_j(f,x)-S|^p\underset{n\to+\infty}{\longrightarrow}0.$$

Similarly, $\mathscr{H}_p$-summability means that

$$\frac{1}{R}\int_0^R|S_r(f,x)-S|^p dr\underset{R\to+\infty}{\longrightarrow}0.$$

In the one-dimensional case, these methods coincide. Moreover, as stated by Hardy and Littlewood, for a function f its continuity at a point $x\in\mathbb{T}$ implies its Fourier series summability in the above sense.

The question about the Fourier series summability by the H_p and $\mathscr{H}_p$ methods in the case of continuous functions in several variables is mainly related to the behavior of the norms

$$H_{n,p}=\sup_{|f|\leqslant 1}\left(\frac{1}{n}\sum_{j=0}^{n-1}|S_j(f,0)|^p\right)^{\frac{1}{p}}\quad\text{and}\quad \mathscr{H}_{R,p}=\sup_{|f|\leqslant 1}\left(\frac{1}{R}\int_0^R|S_r(f,0)|^p dr\right)^{\frac{1}{p}}.$$

In the one-dimensional case, they are bounded, which is equivalent to the Hardy-Littlewood's theorem. In the multidimensional case ($m>1$) the situation is different: for any $p\geqslant 1$ the norms $H_{n,p}$ and $\mathscr{H}_{R,p}$ are unbounded [2-3]. An interesting question is to get an estimation exact in order [4]. Clearly, estimating $\mathscr{H}_{R,p}$ one can regard R to be an integer.

We show that the norms $H_{n,p}$ and $\mathscr{H}_{n,p}$ have the same order. More exactly,[1]

$$H_{n,p}\asymp\mathscr{H}_{n,p}\asymp\begin{cases} n^{\frac{m-1}{2}-\min\{\frac{1}{2},\frac{1}{p}\}} & \text{if}\ \ m\geqslant 3,\ p\geqslant 1;\\ n^{\frac{1}{2}-\frac{1}{p}}\ \min^{\frac{1}{p}}\left\{\ln(n+1),\frac{1}{p-2}\right\} & \text{if}\ \ m=2,\ p>2;\\ \sqrt{\ln(n+1)} & \text{if}\ \ m=2,\ p\in[1,2].\end{cases}$$

Key words and phrases. Multiple Fourier series, spherical sums, strong averages.

[1]We write $\alpha_n\asymp\beta_n$ if $\alpha_n=O(\beta_n)$ and $\beta_n=O(\alpha_n)$. The constants in the inequalities that are true for all n may depend only on the dimension m.

Using standard reasoning, one deduces from these estimations conditions imposed on the continuous function f that ensure the uniform H_p and $\mathscr{H}_p$-summability of the Fourier series, i.e., the uniform (relative to $x \in \mathbb{R}^m$) convergence to zero of the quantities

$$\frac{1}{n}\sum_{j=0}^{n-1} |S_j(f,x) - f(x)|^p \quad \text{and} \quad \frac{1}{R}\int_0^R |S_r(f,x) - f(x)|^p dr.$$

This condition is better formulated in terms of the notion of the best uniform approximation of the function f, defined by the relation

$$E_R(f) = \min_M \|f - M\|_C, \qquad \text{where } M(x) = \sum_{\|k\|\leqslant R} c_k\, e^{ik\cdot x}.$$

Thus, we obtain the statement: *if the function f continuous in $\mathbb{R}^m$ and 2π-periodic in each variable is such that $\mathscr{H}_{R,p}\, E_R(f) \underset{R\to+\infty}{\longrightarrow} 0$, then*

$$\max_x \frac{1}{n}\sum_{j=0}^{n-1} |S_j(f,x) - f(x)|^p \underset{n\to+\infty}{\longrightarrow} 0 \quad \text{and} \quad \max_x \frac{1}{R}\int_0^R |S_r(f,x) - f(x)|^p dr \underset{R\to+\infty}{\longrightarrow} 0.$$

References

[1] Hardy G.H., Littlewood J.E., *Sur la série de Fourier d'une fonction á carré summable,* CR, **156** (1913), 1303-1309.

[2] Kuznetsova O.I., *Strong spherical averages and L-convergence of multiple trigonometric series*, Dokl. Akad. Nauk, **391**:3 (2003), 303–305.

[3] Kuznetsova O.I., *Strong spherical averages of multiple trigonometric series*, Izv. Nat. Akad. Nauk Armenii, **44**:4 (2009), 27–40.

[4] Kuznetsova O.I., Podkorytov A.N., *On strong spherical averages of Fourier series*, Algebra i Analiz, **25**:3, (2013).

Institute for Applied Mathematics and Mechanics, NAS Ukraine
E-mail address: kuznets@iamm.ac.donetsk.ua

St. Petersburg State University
E-mail address: a.podkorytov@gmail.com