FOURIER TRANSFORM VERSUS HILBERT TRANSFORM

ELIJAH LIFLYAND

We present several results in which the interplay between the Fourier transform and the Hilbert transform is of special form and importance. Known relations between the two operators are supplied with new ones.

1. In 50-s (Kahane, Izumi-Tsuchikura, Boas, etc.; see [2]), the following problem in Fourier Analysis attracted much attention:

Let $\{a_k\}_{k=0}^{\infty}$ be the sequence of the Fourier coefficients of the absolutely convergent sine (cosine) Fourier series of a function $f : \mathbb{T} = [-\pi, \pi) \to \mathbb{C}$, that is $\sum |a_k| < \infty$. Under which conditions on $\{a_k\}$ the re-expansion of $f(t)$ ($f(t) - f(0)$, respectively) in the cosine (sine) Fourier series will also be absolutely convergent?

We solve a similar problem for functions on the whole axis and their Fourier transforms. Generally, the re-expansion of a function with integrable cosine (sine) Fourier transform in the sine (cosine) Fourier transform is integrable if and only if not only the initial Fourier transform is integrable but also the Hilbert transform of the initial Fourier transform is integrable.

We observe that a similar answer is true for Fourier series in terms of the discrete Hilbert transform, contrary to the known results that were just sufficient conditions for the summability of the discrete Hilbert transform.

Comparing these two settings, one arrives to the necessity of obtaining effective sufficient conditions for the integrability of the Hilbert transform. Known conditions are discussed and new ones are obtained.

2. The following result is due to Hardy and Littlewood (see, e.g., [4]): *If a (periodic) function f and its conjugate $\widetilde{f}$ are both of bounded variation, their Fourier series converge absolutely.*

We generalize the Hardy-Littlewood theorem (joint work with U. Stadtmüller) to functions on the real axis, hence the absolute convergence of the Fourier series should be replaced by the integrability of the Fourier transform.

Since a function f of bounded variation may be not integrable, its Hilbert transform, a usual substitute for the conjugate function, may not exist. One has to use the modified Hilbert transform

$$\widetilde{f}(x) = \text{(P.V.)}\, \frac{1}{\pi} \int_{\mathbb{R}} f(t) \left\{ \frac{1}{x-t} + \frac{t}{1+t^2} \right\} dt.$$

Theorem 2.1. *Let f be a function of bounded variation and vanish at infinity: $\lim_{|t|\to\infty} f(t) = 0$. If its conjugate $\widetilde{f}$ is also of bounded variation, then the Fourier transforms of both functions are integrable on* $\mathbb{R}$.

The initial Hardy-Littlewood theorem is a partial case of this extension, when the function is taken to be of compact support.

A corresponding extension to radial functions readily follows from the obtained one-dimensional results.

3. In various of the considered problems, one of the main tools is the well-known extension of Hardy's inequality:

$$\int_{\mathbb{R}} \frac{|\widehat{g}(x)|}{|x|}\,dx \lesssim \|g\|_{H^1(\mathbb{R})}.$$

It turns out that if the left-hand side is finite, then g is the derivative of a function of bounded variation f which is locally absolutely continuous, $\lim_{|t|\to\infty} f(t) = 0$, and its Fourier transform is integrable.

This is one of the strongest justifications for systematic investigation of the Fourier transform of a function of bounded variation.

We have found the maximal space for the integrability of such a Fourier transform, it is inspired by work [1]). To be more precise, we let the (integrable) derivative of the given function belong to the class of integrable functions g for which the left-hand side in Hardy's inequality is integrable as well.

Along with those known earlier (see, e.g., [3]), various interesting new spaces appear in this study. Their inter-relations lead, in particular, to improvements of Hardy's inequality.

References

[1] R.L. Johnson and C.R. Warner, *The convolution algebra $H^1(R)$*, J. Function Spaces Appl. **8** (2010), 167–179.

[2] J.-P. Kahane, *Séries de Fourier absolument convergentes*, Springer, Berlin, 1970.

[3] E. Liflyand, *Fourier transforms of functions from certain classes*, Anal. Math. **19** (1993), 151-168.

[4] A. Zygmund, *Trigonometric series*, Vol. I, II, Cambridge Univ. Press, Cambridge, U.K., 1966.

Department of Mathematics, Bar-Ilan University, 5290002 Ramat-Gan, Israel

E-mail address: liflyand@gmail.com