"ALMOST" CAUCHY PROPERTY FOR THE SEQUENCE OF PARTIAL SUMS OF FOURIER SERIES OF FUNCTIONS IN L_p, $p > 1$

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Let 2π-periodic (in each argument) function $f \in L_1(\mathbb{T}^N)$, $\mathbb{T}^N = [-\pi, \pi)^N$, $N \geq 1$, be expanded in multiple trigonometric Fourier series $f(x) \sim \sum c_k e^{ixk}$, and $S_n(x; f)$, $n \in \mathbb{Z}_+^N$, is a rectangular partial sum of this series, and let function $g \in L_1(\mathbb{R}^N)$, $N \geq 1$, be expanded in multiple Fourier integral $g(x) \sim \int \hat{g}(\xi) e^{ix\xi} d\xi$, and $J_\alpha(x; g)$, $\alpha \in \mathbb{R}_+^N$, is a proper Fourier integral.

Suppose that $g(x) = f(x)$ for $x \in \mathbb{T}^N$. Let us define by $R_\alpha(x; f, g)$ the difference $R_\alpha(x; f, g) = S_n(x; f) - J_\alpha(x; g)$, and by $R_\alpha(x; f)$ the difference $R_\alpha(x; f) = S_n(x; f) - J_\alpha(x; g)$, if $g(x) = 0$ out of $\mathbb{T}^N$, where $n = [\alpha] = ([\alpha_1], \ldots, [\alpha_N]) \in \mathbb{Z}_+^N$ ($[t]$ is the integral part of $t \in \mathbb{R}_+^1$). In [1] it was proved that for $N = 2$ and $p > 1$ $R_\alpha(x; f, g) \to 0$ as $\alpha \to \infty$ (i.e. $\min\limits_{1 \leq s \leq N} \alpha_s \to \infty$) almost everywhere on $\mathbb{T}^2$. In the same paper it was proved that conditions $N = 2$, $p > 1$ are essential. In particular, the function $f_0 \in \mathbb{C}(\mathbb{T}^N)$, $N \geq 3$, was constructed such that $\overline{\lim\limits_{\alpha \to \infty}} |R_\alpha(x; f_0)| = +\infty$ everywhere inside $\mathbb{T}^N$. The question arises: how the difference $R_\alpha(x; f, g)$ behaves if the components n_j and α_j of vectors $n \in \mathbb{Z}_+^N$ and $\alpha \in \mathbb{R}_+^N$ are connected by relation:

$$| \, \alpha_j - n_j \, | \leq const, \quad j = 1, \ldots, N. \tag{1}$$

The following Theorem answers this question for $N = 2$.

Theorem 1. *For any $\alpha = (\alpha_1, \alpha_2)$, $\alpha \in \mathbb{R}_+^2$, satisfying condition (1), and for any functions $g(x)$ and $f(x)$ such that $g \in L_p(\mathbb{R}^2)$, $f \in L_p(\mathbb{T}^2)$, $p > 1$, and $g(x) = f(x)$ for $x \in \mathbb{T}^2$,*

$$\lim_{\alpha_1, \alpha_2 \to \infty} R_{\alpha_1, \alpha_2}(x; f, g) = 0 \ \textit{almost everywhere on } \mathbb{T}^2.$$

Further, let us define $RS_{n+m}(x; f) = S_{n+m}(x; f) - S_n(x; f)$, $n, m \in \mathbb{Z}_+^N$. The following result is equivalent to Theorem 1.

Theorem 2. *For any bounded sequence $\{m(n)\}$, $m(n) \in \mathbb{Z}_+^2$, $n \in \mathbb{Z}_+^2$, and for any function $f \in L_p(\mathbb{T}^2)$, $p > 1$,*

$$\lim_{n \to \infty} RS_{n+m(n)}(x; f) = 0 \ \textit{almost everywhere on } \mathbb{T}^2.$$

Note that this estimate is true for the divergent a.e. Fourier series as well.

As we have said above, for $N \geq 3$ Theorem 1 is not true even in $\mathbb{C}(\mathbb{T}^N)$. It is not difficult to prove that result of Theorem 2 is also not true in this class. In this case, the following question arises: how the differences $R_\alpha(x; f, g)$ and $RS_{n+m}(x; f)$ in the classes L_p, for $N \geq 3$, behave if some of the components n_j of vector $n = [\alpha]$ are elements of (single) lacunary sequences ($\{k^{(s)}\}, k^{(s)} \in \mathbb{Z}^1_+$, is a lacunary sequence if $k^{(s+1)}/k^{(s)} \geq q > 1$, $s = 1, 2, \ldots$).

Possibility to obtain new results in the case of additional restrictions on the vector $n = [\alpha]$ is connected with the fact that in the classes L_p, $p > 1$, the "lacunary" subsequences of partial sums of multiple Fourier series have better properties of convergence a.e. in comparison with the whole sequence $S_n(x; f)$.

The partial answers on the latter question are the following results (which, in particular, show that for $N \geq 3$ the differences $R_\alpha(x; f, g)$ and $RS_{n+m}(x; f)$ are not equivalent).

Theorem 3. *There exists a function $f \in \mathbb{C}(\mathbb{T}^N)$, $N \geq 3$, such that for any sequence $\widetilde{\alpha} = (\alpha_3, \ldots, \alpha_N) \in \mathbb{R}^{N-2}_+$*

$$\varlimsup_{n_1, n_2, \widetilde{\alpha} \to \infty} |R_{n_1, n_2, \widetilde{\alpha}}(x; f)| = +\infty \ \textit{ everywhere inside } \mathbb{T}^N.$$

Theorem 4. *For any bounded sequence $\{m(n)\}$, $m(n) \in \mathbb{Z}^2_+$, $n = (n_1, n_2) \in \mathbb{Z}^2_+$, for any lacunary sequences $\{n_j^{(\lambda_j)}\}$, $n_j^{(\lambda_j)} \in \mathbb{Z}^1_+$, $\lambda_j = 1, 2, \ldots$, $j = 3, \ldots, N$, and for any function $f \in L_p(\mathbb{T}^N)$, $p > 1$, $N \geq 3$, almost everywhere on $\mathbb{T}^N$*

$$\lim_{n_1, n_2, \lambda_3, \ldots, \lambda_N \to \infty} RS_{n_1+m_1(n), n_2+m_2(n), n_3^{(\lambda_3)}, \ldots, n_N^{(\lambda_N)}}(x; f) = 0.$$

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References

[1] I.L.Bloshanskii, *On equiconvergence of expansions in multiple trigonometric Fourier series and Fourier integral*, Matem. zametki, **18**:2 (1975), 153–168.

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