REMARKS ON RANK FUNCTIONS AND RANK VARIETIES

MARCIN SKRZYŃSKI

A function $\rho : \mathbb{N} \longrightarrow \mathbb{N}$ is said to be a rank function, if it is weakly decreasing and such that

$$\forall\, j \in \mathbb{N} \setminus \{0\} :\ \rho(j-1) + \rho(j+1) \geq 2\rho(j).$$

Let $\mathcal{M}_n(\mathbb{F})$ be the vector space of all the $n \times n$ matrices over a field $\mathbb{F}$. One can prove that $\rho : \mathbb{N} \longrightarrow \mathbb{N}$ is a rank function if and only if

$$\exists\, A \in \mathcal{M}_{\rho(0)}(\mathbb{F})\, \forall\, j \in \mathbb{N} :\ \operatorname{rank}(A^j) = \rho(j).$$

(In the sequel we write $r_A(j)$ instead of $\operatorname{rank}(A^j)$). The pointwise inequality is a partial order on the set of all the rank function.

It can be shown that if the Zariski closure of a set $\mathcal{E} \subseteq \mathcal{M}_n(\mathbb{F})$ is irreducible, then the set of rank functions $\{r_A :\ A \in \mathcal{E}\}$ has the greatest element [3]. If ρ is a rank function, then

$$\mathcal{X}_\rho = \{A \in \mathcal{M}_{\rho(0)}(\mathbb{F}) :\ r_A \leq \rho\}$$

is an algebraic set of matrices, referred to as a rank variety [1]. Rank functions also appear in the Gerstenhaber-Hesselink theorem on the closure of a nilpotent orbit.

In the talk, we will present some new and some older results on rank functions and their applications in matrix theory and algebraic geometry.

References

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Institute of Mathematics, Cracow University of Technology, ul. Warszawska 24, 31-155 Kraków, Poland

E-mail address: pfskrzyn@cyf-kr.edu.pl