Around Sampling Theorem

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The well-known sampling theorem states that

$$f(x) = \sum_{n\in\mathbb{Z}} f(2^{-j}n)\, \frac{\sin \pi(2^j x - n)}{\pi(2^j x - n)} \tag{1}$$

for any function $f \in L_2(\mathbb{R})$ whose Fourier transform is supported on $[-2^{j-1}, 2^{j-1}]$. From the point of view of wavelet theory, (1) is not a theorem, it is just an illustration for the Shannon MRA. Indeed, the function $\varphi(x) = \frac{\sin \pi x}{\pi x}$ is a scaling function for this MRA, and a function f belongs to the sample space V_j if and only if its Fourier transform is supported on $[-2^{j-1}, 2^{j-1}]$. So, such a function f can be expanded as $f = \sum_{n\in\mathbb{Z}} \langle f, \varphi_{jn}\rangle \varphi_{jn}$, where $\varphi_{jn}(x) = 2^{j/2}\varphi(2^j x + n)$, which coincides with (1). Also, since $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA, any $f \in L_2(\mathbb{R})$ can be represented as

$$f = \lim_{j\to+\infty} \sum_{n\in\mathbb{Z}} \langle f, \varphi_{jn}\rangle \varphi_{jn}. \tag{2}$$

Moreover, (2) has an arbitrary large approximation order. This happens because the function $\varphi(x) = \frac{\sin \pi x}{\pi x}$ is band-limited, a similar property cannot be valid for other natural classes of φ, in particular, for compactly supported φ.

We study operators $Q_j f = \sum_{n\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jn}\rangle \varphi_{jn}$ for a class of band-limited functions φ and a wide class of tempered distributions $\widetilde{\varphi}$. Convergence of $Q_j f$ to f as $j \to +\infty$ in L_2-norm is proved under a very mild assumption on φ, $\widetilde{\varphi}$, and the rate of convergence is equal to the order of Strang-Fix condition for φ. To study convergence in L_p, $p > 1$, we assume that there exists $\delta \in (0, 1/2)$ such that $\overline{\widehat{\varphi}}\widehat{\widetilde{\varphi}} = 1$ a. e. on $[-\delta, \delta]$, $\widehat{\varphi} = 0$ a.e. on $[l-\delta, l+\delta]$ for all $l \in \mathbb{Z} \setminus \{0\}$. For appropriate band-limited or compactly supported functions $\widetilde{\varphi}$, the estimate $\|f - Q_j f\|_p \le C\omega_r(f, 2^{-j})_{L_p}$, where ω_r denotes the r-th modulus of continuity, is obtained for arbitrary $r \in \mathbb{N}$. For tempered distributions $\widetilde{\varphi}$, we prove that $Q_j f$ tends to f, $f \in S$, in L_p-norm, $p \ge 2$, with an arbitrary large approximation order. In particular, for some class of differential operators L, we consider $\widetilde{\varphi}$ such that $Q_j f = \sum_{n\in\mathbb{Z}} Lf(2^{-j}\cdot)(n)\varphi_{jn}$. The corresponding wavelet frame-type expansions are found.