EXTENSIONS OF SCHUR'S INEQUALITY FOR THE LEADING COEFFICIENT OF BOUNDED POLYNOMIALS WITH ONE PRESCRIBED ZERO

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Our point of departure is Schur's Chebyshev-type inequality

$$|a_n| \leq (\cos\frac{\pi}{4n})^{2n}2^{n-1}, \tag{1}$$

cf. [3, (16.4.6)] and [5, Theorem III*], for the leading coefficient of $P_n \in \mathbf{B_{n,0}} = \{P_n : ||P_n||_{\infty,\mathbf{I}} \leq 1 \text{ and } P_n(-1) = 0\}$, where $P_n(x) = \sum_{k=0}^{n} a_k x^k$ and $||\cdot||_{\infty,\mathbf{I}}$ denotes the uniform norm on $\mathbf{I} = [-1, 1]$. Let $\mathbf{C_{n,0}}$ denote the larger set of P_n's, $n \geq 2$, which are bounded by 1 only at the $n+1$ extremal points of T_n (Chebyshev polynomial on $\mathbf{I}$) and furthermore satisfy the asymmetric boundary condition $P_n(-1) = 0$ (the case $P_n(1) = 0$ runs analogously). Our results include:

(I) Sharp V.A. Markov-type inequalities, cf. [3, Theorems 16.3.1; 16.3.2], for all coefficients of $P_n \in \mathbf{C_{n,0}}$, and we determine the extremal polynomials (i.e., those for which equality is attained) within $\mathbf{C_{n,0}}$. In particular, the leading coefficient of P_n obeys

$$|a_n| \leq (1 - \frac{1}{2n})2^{n-1}. \tag{2}$$

This is in contrast to the unconstrained case (i.e., no prescribed zero on $\mathbf{I}$), where both (1) and (2) would coincide with Chebyshev's inequality $|a_n| \leq 2^{n-1}$, cf. [3, (16.3.2); (16.3.4)].

(II) Sharp Szegö-type inequalities for all consecutive pairs $|a_{k-1}|+|a_k|$ of coefficients of $P_n \in \mathbf{C_{n,0}}$, with $n-k$ even, and we determine the extremal polynomials within $\mathbf{C_{n,0}}$. This result is curious because the sharp upper bounds turn out to be the moduli of the corresponding coefficients of T_n (as in Szegö's inequality [3, Theorem 16.3.3]), although T_n is not a member of $\mathbf{C_{n,0}}$ since $T_n(-1) \neq 0$. In particular, the leading pair obeys

$$|a_{n-1}| + |a_n| \leq 2^{n-1}. \tag{3}$$

(III) Sharp Schur-type inequalities for all a_k, with $n-k$ even, if $P_n \in \mathbf{B_{n,0}}$. In particular,

$$|a_{n-2}| \leq (1-2(n-1)(\sin\frac{\pi}{4n})^4)(\cos\frac{\pi}{4n})^{2(n-2)}n2^{(n-2)-1}. \tag{4}$$

The corresponding estimate for a_{n-2} (according to (I)), if $P_n \in \mathbf{C_{n,0}}$, is less involved:

$$|a_{n-2}| \leq (1-\frac{n-2}{2n^2})n2^{(n-2)-1}. \tag{5}$$

A generalization to $P_n(-1) = \gamma \neq 0$ is possible. For alternative extensions of (1) see [1], [4]. Extensions of Schur's second Chebyshev-type coefficient inequality [5, Theorem IV*] for polynomials which vanish symmetrically at both endpoints of $\mathbf{I}$ are given in [2].

References

[1] M.A. Qazi & Q.I. Rahman, *On a polynomial inequality of P.L. Chebyshev*, Arch. Inequal. Appl. **1** (2003), 31–41.

[2] H.-J. Rack, *Extensions of Schur's inequality for the leading coefficient of bounded polynomials with two prescribed zeros*, pp. 107–116. In: Advances in Applied Mathematics and Approximation Theory (G.A. Anastassiou and O. Duman, Eds.), Springer Proceedings in Mathematics & Statistics **41**, Springer, New York, 2013.

[3] Q.I. Rahman & G. Schmeisser, *Analytic Theory of Polynomials*, London Mathematical Society Monographs N. S. **26**, Clarendon Press, Oxford, 2002.

[4] Q.I. Rahman & G. Schmeisser, *Inequalities for polynomials on the unit interval*, Trans. Am. Math. Soc. **231** (1977), 93–100.

[5] I. Schur, *Über das Maximum des absoluten Betrages eines Polynoms in einem gegebenen Intervall*, Math. Zeitschr. **4** (1919), 271–287.

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