ALGEBRAIC COSINE VALUES AT RATIONAL MULTIPLES OF π

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Rational and higher algebraic values of the cosine function have been of much interest for quite some time, cf. [3, 2, 5, 6]. As early as 1933, D. H. Lehmer, [3], proved that if $k/n,\ n > 2$, is an irreducible fraction, then $2\cos(2\pi k/n)$ is an algebraic integer of degree $\varphi(n)/2$, where $\varphi(n)$ is the Euler's totient function. Lehmer's proof makes use of cyclotomic polynomials. As a consequence, we have ([4, Theorem 6.16, pp. 308-309]): let $\theta = r\pi$ be a rational multiple of π. Then $\cos\theta$ is irrational except when $\cos\theta = 0, \pm 1/2, \pm 1$. Recently, Varona, [6], proved that if $r \in \mathbb{Q} \cap [0, 1]$, then $\arccos(\sqrt{r})$ is a rational multiple of π if and only if $r \in \{0, 1/4, 1/2, 3/4, 1\}$. His proof is elementary and is similar to the proof of [1, Theorem 4, p. 32].

Our objectives here are:

- to use elementary trigonometric identities to find all nonnegative rational and and some quadratic values of the cosine function at rational multiples of π extending the result in [6];
- to use Lehmer's results in [3] to determine all algebraic cosine values at rational multiples of π;
- to explicitly work out all algebraic cosine values of degree less than 5 at rational multiples of π.

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